

COHOMOLOGY RINGS OF GOOD CONTACT TORIC MANIFOLDS

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ABSTRACT. A good contact toric manifold M is determined by its moment cone C . We compute the equivariant cohomology ring with \mathbb{Z} coefficient of M in terms of the combinatorial data of C . Then under a smoothness criterion on the cone C , we compute the singular cohomology ring with \mathbb{Z} coefficient of M in terms of the combinatorial data of C .

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1. Introduction

1.1. *Good contact toric manifolds.* A *contact toric manifold* of dimension $2n - 1$ is a compact connected $(2n - 1)$ -dimensional contact manifold equipped with an effective Hamiltonian action of an n -torus T^n . One can find an introduction to Hamiltonian actions on contact manifolds and definitions of related notions in [L1]. We will not

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use these concepts in any essential way in this paper. Motivated by the work of Banyaga and Molino [BnM1],[BnM2],[Bn], and of Boyer and Galicki [BoyG], Lerman gave a full-classification theorem of contact toric manifolds in [L1]. According to Lerman's theorem, when $n \geq 3$ and the torus action is not free, the contact toric manifolds are classified by their *moment cones*, which by definition are the union of $\{\text{origin}\}$ and the moment map image of their symplectizations. These moment cones are all *good cones*.

Definition 1.1. A **good cone** in \mathbb{R}^n is a rational polyhedral cone given by

$$(1.1) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\},$$

where $F_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle = 0\}$ is a facet of C and $v_i \in \mathbb{Z}^n$ is the inward-pointing primitive normal vector to F_i . In addition this cone must satisfy:

(i) For $0 < l < n$, each codimension l face F of C is contained in exactly l facets:

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l};$$

and

(ii) The \mathbb{Z} -module generated by v_{i_1}, \dots, v_{i_l} is a direct summand of \mathbb{Z}^n of rank l .

Definition 1.2. A contact toric manifold M is called a **good contact toric manifold** if $\dim M > 3$ and the moment cone of M is a strictly convex good cone.

This explains the title of this paper.

Remark 1.3. The requirement that the moment cone being strictly convex is equivalent to asking M to be of *Reeb type*.

Remark 1.4. When the moment cone is not strictly convex, the construction of contact toric manifold with the given moment cone can be found in the proof of Proposition 4.7 in [LS]. They are topologically $T^k \times S^{k+2l-1}$.

1.2. Equivariant cohomology. Assume G is a Lie group and M is a topological space with a G -action. Let EG be a contractible topological space with a free G -action. Then G acts freely on $EG \times M$ diagonally. The quotient space of this action, which we will denote by $(EG \times M)/G$ and $EG \times_G M$ interchangeably, is called the *Borel construction* of the G -space M . The homotopy type of $EG \times_G M$ is independent of the choice of EG . Notations such as M_G and $EG \times^G M$ are also used in literature to denote the Borel construction of M . The *equivariant cohomology* of M is defined as the ordinary cohomology of $EG \times_G M$ and is denoted $H_G^*(M)$.

The equivariant cohomology of a point with trivial torus action is of particular interest to us in this paper. The Borel construction of a point $ET^m \times_{T^m} pt = ET^m/T^m$ is the classifying space of T^m , denoted by BT^m . If we use $(S^\infty)^m$ as the

model for ET^m , then BT^m is the product of m copies of $\mathbb{C}P^\infty$. The equivariant cohomology of a point is thus a polynomial ring with m variables, i.e,

$$(1.2) \quad H_{T^m}^*(pt; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m],$$

where $x_i \in H_{T^m}^2(pt; \mathbb{Z})$.

The variable x_i is the first Chern class of the fiber bundle

$$(1.3) \quad ET^m \times_{T^m} \mathbb{C} \rightarrow ET^m \times_{T^m} pt,$$

where in the total space, T^m acts on \mathbb{C} with weight $-(0, 0, \dots, 1, \dots, 0)$, where the 1 is in the i^{th} position.

The generators for the various cohomology rings in this paper will always be the images of these x_i 's under certain maps.

It is not hard to see that equivariant cohomology is an equivariant homeomorphism invariant. In this paper, we are concerned with equivariant and ordinary cohomology rings. Since both of these are equivariant homeomorphism invariants, two T^n -spaces that are equivariantly homeomorphic are considered as identical spaces. Because of this, we may exploit an idea of Davis and Januskiwicz in [DJ], where they define a topological counterpart for toric manifolds and compute the corresponding cohomology rings.

1.3. Outline of the paper. In Section 2, we imitate the construction given in [DJ] to define several topological spaces with torus actions, using combinatorial methods. These spaces will be shown in later sections to be equivariantly homeomorphic to toric symplectic cones, good contact toric manifolds and symplectic toric orbifolds respectively.

In Section 3, exploiting the techniques in [DJ] with some modification, we compute the equivariant cohomology of a good contact toric manifold. The main theorem is Theorem 3.10.

In Section 4, we collect some facts about symplectic toric manifolds and setup some notation for Section 5.

In Section 5, we compute the ordinary cohomology of a good contact toric manifold. The work in this section is not carried out in full generality. We must first assume C_0 , the cone minus the origin, is contained in the upper half space

$$(1.4) \quad U\mathbb{R}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

We will show that this assumption will not cause any loss of generality.

The intersection of C with the hyperplane

$$(1.5) \quad H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\}$$

is a simple rational convex polytope. We denote it by P . Assume $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$ is the primitive inward-pointing normal vector to F_i , the i^{th} facet of the cone. We need to impose the following criterion on C .

Smoothness Criterion: The polytope P is a Delzant polytope in H , and the vector $(v_{i1}, v_{i2}, \dots, v_{i,n-1})$ is primitive in \mathbb{Z}^{n-1} .

This hypothesis is where the generality is lost. We call this a *smoothness criterion*, since a Delzant polytope is by definition a smooth simple rational convex polytope. We make this hypothesis so we do not need to deal with orbi-bundles over orbifolds.

Under this smoothness criterion, we show that M , a good contact toric manifold, is a principle S^1 -bundle over a symplectic toric manifold N . Then using the Gysin sequence of the S^1 -bundle, we compute the ordinary cohomology group of M and also show how to take the product of any two even degree cohomology classes, and the product of one even degree cohomology class and one odd degree cohomology class. This is Theorem 5.13. Then by relating the Euler class of the S^1 -bundle to the symplectic form on N and using the Hard Lefschetz Theorem, we show that the odd degree cohomology of M vanishes in degree lower than half, and hence show the product of any two odd degree cohomology classes of M is zero for dimension reasons. This is Theorem 5.14.

Finally in the Appendix, we give several equivalent descriptions of the generators of the cohomology ring of a symplectic toric manifold.

1.4. Relation with other work. Some of the geometric computations we describe in this paper have been computed in a more algebraic fashion by other authors. The virtue of our computation is that it is more explicit and geometric. Moreover, the geometry allows us to use the Hard Lefschetz Theorem on symplectic toric manifold to deduce Theorem 5.15. The consequent vanishing of certain Betti numbers is much more obscure in the existing algebraic description of the cohomology ring. The integral equivariant cohomology ring of a general smooth toric variety, which includes the symplectization of good contact toric manifolds, was identified with the Stanley-Reisner ring earlier by Franz [Fr, Sec. 3] with a different proof from our proof of Theorem 3.10. In Theorem 1.2 of the same paper of Franz, he got an expression of the integral ordinary cohomology of a smooth toric variety in terms of Tor modules. The ordinary cohomology ring of the quotient of a moment-angle complex, which includes the case we consider in Section 5, was expressed also using Tor modules in [BP, Thm.7.37]. To relate the description in Theorem 5.13 in this paper to the results of [Fr] and [BP] uses an algebraic argument that will be explained in [LMM].

A remark on notations: If a group G acts on two spaces X and Y , we use $X \times_G Y$ and $(X \times Y)/G$ interchangeably to denote the quotient space of $X \times Y$

under the diagonal G action. And we will use $[x, y]$ to denote the element of $X \times_G Y$ that is the equivalence class of $(x, y) \in X \times Y$, so $[gx, gy] = [x, y]$, for $g \in G$.

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2. Basic definitions, constructions and examples

A $2n$ dimensional **symplectic toric manifold** is a compact connected symplectic manifold equipped with an effective Hamiltonian action of an n -torus T^n . Delzant showed in [D] that these geometric objects are classified by their moment image, which is a simple rational smooth polytope, called a **Delzant polytope** in the symplectic literature. More information about symplectic toric manifolds may be found in Chapter 28 of [CdS].

The analogous result for contact toric manifolds was given by Lerman in [L1], where he showed that a large class of contact toric manifolds are classified by **good cones**, defined in Definition 1.1.

The full classification theorem of contact toric manifolds is Theorem 2.18 in [L1]. The terminology **good contact toric manifold**, which is defined in Definition 1.2, used in this paper belongs to the case(4) in Lerman's classification theorem. It is actually a proper subcase, since here we also require the moment cone to be strictly convex.

A **toric symplectic cone** is the symplectization of a contact toric manifold. Toric symplectic cones and contact toric manifolds are in one-to-one correspondence. There is more information about symplectic cones in [L3] and [AM]. We will review the method to obtain toric symplectic cones and contact toric manifolds from strictly convex good cones in Section 3.

The Delzant polytope, which is the moment image of symplectic toric manifold, is in fact the orbit space of the T^n action, and the moment map is just the point to orbit map. Using the ideas of [DJ], with simple combinatorial methods we can construct manifolds that are T^n -equivariantly homeomorphic to symplectic toric manifolds. The constructions we describe below are a variation and generalization of those in [DJ].

We let P (or C_0) be a simple convex polytope (or a strictly convex good cone minus the origin) in \mathbb{R}^n . The set of facets of P (or C_0) is denoted \mathcal{F} , and we write F_i for the i^{th} facet. A **characteristic map** is a map

$$\lambda : \mathcal{F} \rightarrow \mathbb{Z}^l$$

where \mathbb{Z}^l is the integral lattice in \mathbb{R}^l . Here l and n are not necessarily equal. Denote $\lambda(F_i)$ by λ_i .

Every finite subset U of \mathbb{Z}^l determines a subgroup of T^l generated by

$$\{(e^{u_1\theta}, e^{u_2\theta}, \dots, e^{u_l\theta}) : \theta \in \mathbb{R}, (u_1, \dots, u_l) \in U\}$$

It is in fact a closed subgroup. For every point p in P (or C_0), denote by S_p^λ the subgroup of T^l determined by vectors

$$\{\lambda_i : p \in F_i\}.$$

Define an equivalence relation Δ on $T^l \times P$ by:

$$(2.1) \quad (g, p)\Delta(h, q) \Leftrightarrow p = q, \text{ and } g^{-1}h \in S_p^\lambda.$$

We say Δ is the equivalence relation determined by λ . Let Δ_p denote S_p^λ , and P^λ denote the quotient space $(T^l \times P)/\Delta$. With the quotient topology and the T^l action on the first coordinate by multiplication, P^λ is in the category of T^l -topological spaces.

We now give three fundamental examples which will be heavily used throughout the paper.

Example 2.1. Let

$$(2.2) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

be a strictly convex good cone in \mathbb{R}^n . The vector v_i is the inward-pointing primitive normal vector to F_i . We assume the equations are **minimal**; that is, removing any one of the equations will give a different set. We will always assume that the equations used to define convex polytopes or cones are minimal. We let $C_0 = C \setminus \{\text{origin}\}$. We set $l = n$ and define a characteristic map λ by $\lambda(F_i) = v_i$. We denote by Δ the equivalence relation on $T^n \times C_0$ determined by λ . Then $C_0^\lambda = (T^n \times C_0)/\Delta$, with T^n acting on the first coordinate by multiplication, is T^n -equivariantly homeomorphic to a toric symplectic cone. This claim will be proved in Section 3.

Example 2.2. For C as in the previous example, it is a cone over a simple convex polytope P . Notice that the facets of P are in one-to-one correspondence with those of C_0 . We let \tilde{F}_i denote the facet of P contained in F_i . We may define a characteristic map as in Example 2.1 by sending \tilde{F}_i to v_i . By abuse of notation, we still call this map λ and denote by Δ the equivalence relation on $(T^n \times P)$ determined by λ . We emphasize that $\lambda_i = \lambda(\tilde{F}_i)$ is the normal vector to F_i , not to \tilde{F}_i in P . Then $P^\lambda = (T^n \times P)/\Delta$, with T^n acting on the first coordinate by multiplication, is T^n -equivariantly homeomorphic to the contact toric manifold M associated to C . The space C_0^λ in the previous example is the symplectization of $M = P^\lambda$. These claims will be proved in Section 3.

The last example is related to the polytope P itself, without reference to the cone C .

Example 2.3. Let P be a Delzant polytope, i.e. a convex simple rational smooth polytope, in \mathbb{R}^k given by

$$(2.3) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

The equations are again assumed to be minimal, as in Example 2.1. We denote the i^{th} facet by \tilde{F}_i . Then \tilde{v}_i is the inward-pointing primitive normal vector to the facet \tilde{F}_i . We set $l = k$ and define a characteristic map $\tilde{\lambda}$ by setting $\tilde{\lambda}(\tilde{F}_i) = \tilde{v}_i$. Denote by $\tilde{\Delta}$ the equivalence relation on $T^k \times P$ determined by $\tilde{\lambda}$. Then $P^{\tilde{\lambda}} = (T^k \times P)/\tilde{\Delta}$, with T^k acting on the first coordinate by multiplication, is T^k -equivariantly homeomorphic to the symplectic toric manifold associated to P . This will be restated in Section 4.

Notice the subtle difference between P^λ and $P^{\tilde{\lambda}}$. Their relation is crucial in this paper.

3. Equivariant cohomology of good contact toric manifolds

In this section, we imitate the computation of the equivariant cohomology of a symplectic toric manifold given in [DJ] to compute the equivariant cohomology of a good contact toric manifold.

Remark 3.1. In their paper [DJ], Davis and Januszkiewicz defined a class of manifolds equipped with torus action which they called *toric manifolds*, now called *quasitoric manifolds*, and computed their equivariant and ordinary cohomology rings. The cohomology ring of symplectic toric manifolds is a particular example of the cohomology ring of a quasitoric manifold. Quasitoric manifolds are a strictly larger class than symplectic toric manifolds. More information may be found in [GP]. Constructions in [DJ] were later generalized to orbifolds in [PS].

We begin with a brief review of the construction of a toric symplectic cone from a strictly convex good cone given in Lemma 6.4 in [L1]. Let the cone be as in Example 2.1:

$$(3.1) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

Define a map

$$(3.2) \quad \pi_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$$

by sending the i^{th} standard basis vector e_i of \mathbb{Z}^m to v_i . This induces a map

$$(3.3) \quad \pi_{\mathbb{R}} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

by tensoring with \mathbb{R} . We then get a map

$$(3.4) \quad \pi_T : T^m = \mathbb{R}^m / \mathbb{Z}^m \rightarrow T^n = \mathbb{R}^n / \mathbb{Z}^n$$

The subscripts $\mathbb{Z}, \mathbb{R}, T$ may be omitted when it will not cause confusion.

Let $K = \ker(\pi_T)$. Then we have a short exact sequence of groups

$$(3.5) \quad 1 \longrightarrow K \xrightarrow{i} T^m \xrightarrow{\pi_T} T^n \longrightarrow 1.$$

This induces maps between Lie algebras

$$(3.6) \quad 0 \longrightarrow \mathfrak{k} \xrightarrow{i_*} \mathfrak{t}^m \xrightarrow{\pi_*} \mathfrak{t}^n \longrightarrow 0,$$

with dual maps between the duals of the Lie algebras

$$(3.7) \quad 0 \longleftarrow \mathfrak{k}^* \xleftarrow{i^*} (\mathfrak{t}^m)^* \xleftarrow{\pi^*} (\mathfrak{t}^n)^* \longleftarrow 0.$$

Let $u : \mathbb{C}^m \rightarrow (\mathfrak{t}^m)^*$ be defined by

$$(3.8) \quad u(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2).$$

Lerman showed in [L1] that

$$(3.9) \quad S = ((i^* \circ u)^{-1}(0) \setminus \{0\}) / K$$

is the toric symplectic cone associated to C .

The standard action of T^m on \mathbb{C}^m restricts to a T^m -action on $(i^* \circ u)^{-1}(0) \setminus \{0\}$, and hence a K -action on $(i^* \circ u)^{-1}(0) \setminus \{0\}$. It induces a $T^n \cong T^m / K$ action on $S = ((i^* \circ u)^{-1}(0) \setminus \{0\}) / K$.

In other words, the action of $t_n \in T^n$ on S is induced by the standard action of any element in $\pi^{-1}(t_n)$ on $(i^* \circ u)^{-1}(0) \setminus \{0\}$. This action is Hamiltonian with moment map

$$(3.10) \quad \nu : S \rightarrow (\mathfrak{t}^n)^*$$

$$(3.11) \quad [z_1, \dots, z_m] \mapsto (\pi^*)^{-1}(u(z_1, \dots, z_m)),$$

using that π^* is injective. The image of ν is $C_0 = C \setminus \{0\}$ according to [L1]. It is not hard to see from the construction that C_0 is the orbit space of the T^n action on S .

Now we are ready to prove the claim made in Example 2.1 as follows.

Proposition 3.2. *With the same notation as in Example 2.1, C_0^λ is T^n -equivariantly homeomorphic to S , the symplectic toric cone associated with C .*

Proof. If we restrict the domain of u defined in (3.8) to

$$(3.12) \quad (\mathbb{R}^m)^+ = \{(z_1, \dots, z_m) \in \mathbb{C}^m : z_i \in \mathbb{R}_{\geq 0}, \forall i\},$$

it is a homeomorphism onto its image. Call this restriction map u_0 .

Now define a map

$$(3.13) \quad \gamma_0 : T^n \times C_0 \rightarrow S$$

by:

$$(t_n, p) \mapsto t_n \cdot [u_0^{-1}(\pi^*(p))]$$

We note that $\pi^*(p) \in (\mathbb{R}^m)^+$ because of the defining equations for C . Thus, $u_0^{-1}(\pi^*(p))$ is well-defined, and it is straightforward to check that

$$u_0^{-1}(\pi^*(p)) \in (i^* \circ u)^{-1}(0) \setminus \{0\}.$$

We let $[u_0^{-1}(\pi^*(p))] \in S$ be the equivalence class of $u_0^{-1}(\pi^*(p))$. Finally, $t_n \in T^n$ acts on S as noted earlier.

This map γ_0 is a continuous. It is also surjective because C_0 is the orbit space of the T^n -action on S .

According to Lemma 3.3, which will be stated and proved right after this proposition, γ_0 induces a bijective map

$$(3.14) \quad \gamma : (T^n \times C_0)/\Delta \rightarrow S$$

The inverse of this map is also continuous since γ is an open map. Finally, γ is obviously T^n -equivariant. \square

Lemma 3.3. *The stabilizer of $[u_0^{-1}(\pi^*(p))] \in S$ is $\Delta_p \leq T^n$.*

Proof. For simplicity and without loss of generality, we may assume that the facets that contain p are exactly F_1, \dots, F_j . Thus, the coordinates of $\pi^*(p)$ that are zero are exactly the first j coordinates. Consequently, the stabilizer of $[u_0^{-1}(\pi^*(p))]$ must be

$$\{\pi_T(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_j}, 1, 1, \dots, 1) : \theta_1, \dots, \theta_j \in \mathbb{R}\}.$$

This is precisely Δ_p . \square

A few more lines will prove the claims made in Example 2.2.

Proposition 3.4. *Let M denote the good contact toric manifold associated to C . With the same notation as in Example 2.2, P^λ is T^n -equivariantly homeomorphic to M .*

Proof. By definition, the symplectic cone S is the symplectization of M . Topologically, $S = M \times \mathbb{R}$. The map ν defined in (3.10) is proportional on the second coordinate.

For each $q \in M$, there is a unique $x = x(q) \in \mathbb{R}$, such that

$$\nu(q, x(q)) \in P.$$

Notice that $P^\lambda = (T^n \times P)/\Delta$ is a subset of $(T^n \times C_0)/\Delta$, so $(T^n \times P)/\Delta$ is T^n -equivariantly homeomorphic to its image under γ as defined in (3.14). This is precisely the pre-image of P under ν , which is

$$\{(q, x(q)) \in S = M \times \mathbb{R} : q \in M\}.$$

This is clearly T^n -equivariantly homeomorphic to M . \square

We now compute the T^n equivariant cohomology of M , or equivalently, of P^λ . Recall that C is a cone over P . We continue to let λ denote the characteristic map defined in Example 2.2. Suppose $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$. Let $\tilde{\mathcal{F}}$ be the set of facets of P , and $m = |\tilde{\mathcal{F}}|$, the number of facets. Define a characteristic map

$$(3.15) \quad \mu : \tilde{\mathcal{F}} \rightarrow \mathbb{Z}^m$$

by sending \tilde{F}_i to e_i , the i^{th} standard basis vector of \mathbb{Z}^m . Denote by Ω the equivalence relation on $T^m \times P$ determined by μ . This space $P^\mu = (T^m \times P)/\Omega$ was first defined in [DJ].

Lemma 3.5. *P^μ is a fiber bundle over P^λ with fiber K , where K was defined in (3.5).*

Proof. The group K acts naturally on $P^\mu = (T^m \times P)/\Omega$ by multiplication on the first coordinate. Since C is a good cone, $K \cap \Omega_p = 1$ for every $p \in P$. Thus the K action on P^μ is free. The orbit space is exactly $P^\lambda = (T^n \times P)/\Delta$. The projection from the total space to the orbit space is given by the natural map

$$\begin{aligned} \pi' : P^\mu = (T^m \times P)/\Omega &\rightarrow P^\lambda = (T^n \times P)/\Delta \\ [t_m, p] &\mapsto [\pi_T(t_m), p]. \end{aligned}$$

\square

The torus K may be disconnected. Denote by K_0 the connected component of K containing identity.

Lemma 3.6. *There exists a group homomorphism $r : T^m \rightarrow K_0$, satisfying $r \circ i|_{K_0} = id_{K_0}$, where id_{K_0} denotes the identity map on K_0 and i is the inclusion of K into T^m as defined in (3.5).*

Proof. Consider the maps

$$0 \longrightarrow \ker(\pi_{\mathbb{Z}}) \xrightarrow{j} \mathbb{Z}^m \xrightarrow{\pi_{\mathbb{Z}}} \mathbb{Z}^n,$$

where $\pi_{\mathbb{Z}}$ is defined in (3.2) and j is the inclusion map. As a subgroup of the free abelian group \mathbb{Z}^m , $\ker(\pi_{\mathbb{Z}})$ is also a free abelian group. So with suitably chosen basis, the map j looks like an inclusion

$$b_1\mathbb{Z} \oplus b_2\mathbb{Z} \oplus \cdots \oplus b_k\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \mathbb{Z} \oplus \mathbb{Z}^{m-k}.$$

But notice that for every $x \in \mathbb{Z}^m$ and every $t \in \mathbb{Z} \setminus \{0\}$,

$$x \in \ker(\pi_{\mathbb{Z}}) \Leftrightarrow tx \in \ker(\pi_{\mathbb{Z}}).$$

So all of the b_i must be 1. This means $\ker(\pi_{\mathbb{Z}})$ is a direct summand of \mathbb{Z}^m , so there is a group homomorphism

$$(3.16) \quad r_0 : \mathbb{Z}^m \rightarrow \ker(\pi_{\mathbb{Z}}),$$

such that

$$r_0 \circ j = id_{\ker(\pi_{\mathbb{Z}})}.$$

This r_0 induces a map

$$(3.17) \quad r : T^m \rightarrow K_0$$

that satisfies the requirement of the lemma. \square

Using Lemma 3.6, we can define an action of T^m on $EK_0 \times ET^n$ by first sending $t_m \in T^m$ to $(r(t_m), \pi(t_m)) \in K_0 \times T^n$, then using diagonal action of this element on $EK_0 \times ET^n$.

Lemma 3.7. *The space $(EK_0 \times ET^n) \times_{T^m} ((T^m \times P)/\Omega)$ is a fiber bundle over $ET^n \times_{T^n} ((T^n \times P)/\Delta)$ with fiber EK_0 . Thus these two spaces are homotopy equivalent.*

Proof. Define the projection map by

$$[x, y, [t, p]] \mapsto [y, [\pi(t), p]],$$

where $x \in EK_0$, $y \in ET^n$, $t \in T^m$, $p \in P$. The square brackets $[]$ are used whenever there is a equivalence relation involved.

It is easily shown that the map is well-defined. As for the fiber, pick any point $[y_0, [t_n, p]]$ in the base space, with $y_0 \in ET^n$, $t_n \in T^n$, $p \in P$. Assume $[x, y, [t, p]]$ is in the pre-image, then

$$[y, [\pi(t), p]] = [y_0, [t_n, p]].$$

So there exists $s \in T^n$ such that

$$sy = y_0, \quad \text{and}$$

$$[s \cdot \pi(t), p] = [t_n, p].$$

Since π_T is surjective, so there is an $s' \in T^m$ such that $\pi(s') = s$. Then

$$[x, y, [t, p]] = [r(s')x, \pi(s')y, [s't, p]] = [r(s')x, y_0, [s't, p]], \quad \text{and}$$

$$\pi'[(s't), p] = [\pi(s')\pi(t), p] = [s \cdot \pi(t), p] = [t_n, p],$$

where π' is defined in Lemma 3.5.

This means when considering the fiber over $[y_0, [t_n, p]]$, we only need to use representatives of the form $[x, y_0, [t, p]]$, where y_0 is fixed, and $\pi'[t, p] = [t_n, p]$.

By Lemma 3.5, we know that the set of points $[t, p] \in P^\lambda$ that satisfy $\pi'[t, p] = [t_n, p]$ is homeomorphic to K . The elements in T^m that fix y_0 are K . So the fiber over $[y, [t_n, p]]$ is homeomorphic to $(EK_0 \times K)/K$, which is homeomorphic to EK_0 via the map

$$[x, k] \mapsto r(k)^{-1}x,$$

noticing that K acts on EK_0 by first mapping $k \in K$ to $r(k) \in K_0$, then applying $r(k)$ to EK_0 . \square

Notice that the cohomology of $ET^n \times_{T^n} ((T^n \times P)/\Delta)$ is exactly what we want to compute: $H_{T^n}^*(M; \mathbb{Z})$.

Davis and Januskiewicz computed $H_{T^m}^*(P^\mu; \mathbb{Z}) = H^*(ET^m \times_{T^m} ((T^m \times P)/\Omega); \mathbb{Z})$ in [DJ]. More details can be found in [DJ, 434-436]. So the remaining task is to compare the two spaces

$$(EK_0 \times ET^n) \times_{T^m} ((T^m \times P)/\Omega)$$

and

$$ET^m \times_{T^m} ((T^m \times P)/\Omega).$$

Lemma 3.8. *The two spaces*

$$(3.18) \quad (EK_0 \times ET^n) \times_{T^m} ((T^m \times P)/\Omega)$$

and

$$(3.19) \quad ET^m \times_{T^m} ((T^m \times P)/\Omega)$$

are homotopy equivalent.

Proof. Let $W = EK_0 \times ET^n \times ET^m$. An element $t_m \in T^m$ acts on W by the diagonal action of $(r(t_m), \pi(t_m), t_m)$.

The projection

$$(3.20) \quad p_1 : W \times_{T^m} (T^m \times P)/\Omega \rightarrow ET^m \times_{T^m} (T^m \times P)/\Omega$$

is a fiber bundle with fiber $EK_0 \times ET^n$, which is contractible, so

$$(3.21) \quad W \times_{T^m} (T^m \times P)/\Omega \stackrel{h.e.}{\sim} ET^m \times_{T^m} (T^m \times P)/\Omega,$$

where $\stackrel{h.e.}{\sim}$ stands for ‘homotopy equivalent’.

Suppose $t_m \in T^m$. If $t_m \notin K$, then $\pi(t_m) \neq 1$, so t_m acts on $EK_0 \times ET^n$ with no fixed-points. If $t_m \notin \Omega_p$ for any $p \in P$, then t_m acts on $(T^m \times P)/\Omega$ with

no fixed-points. Since $K \cap \Omega_p = 1$ for all $p \in P$, the diagonal action of T^m on $(EK_0 \times ET^n) \times (T^m \times P)/\Omega$ is free.

So the projection

$$(3.22) \quad p_2 : W \times_{T^m} (T^m \times P)/\Omega \rightarrow (EK_0 \times ET^n) \times_{T^m} (T^m \times P)/\Omega$$

is a fiber bundle with fiber ET^m . So

$$(3.23) \quad W \times_{T^m} (T^m \times P)/\Omega \stackrel{h.e.}{\sim} (EK_0 \times ET^n) \times_{T^m} (T^m \times P)/\Omega.$$

Combining (3.21) and (3.23) completes the proof. \square

Definition 3.9. Define \mathcal{I} to be the ideal of $\mathbb{Z}[x_1, x_2, \dots, x_m]$ or $\mathbb{Q}[x_1, x_2, \dots, x_m]$ generated by monomials

$$\left\{ x_{i_1} x_{i_2} \cdots x_{i_l} : \bigcap_{j=1}^l \widetilde{F}_{i_j} = \emptyset \right\}.$$

The coefficients used will be clear in context.

Theorem 3.10. If M is the contact toric manifold associated with the strictly convex good cone

$$C = \bigcap_{i=1}^m \{x \in (t^n)^* = \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

then

$$H_{T^n}^*(M; \mathbb{Z}) \simeq \mathbb{Z}[x_1, \dots, x_m]/\mathcal{I},$$

where $x_i \in H_{T^n}^2(M; \mathbb{Z})$.

Proof. According to Lemma 3.7,

$$H_{T^n}^*(M; \mathbb{Z}) = H^*(ET^n \times_{T^n} P^\lambda, \mathbb{Z}) = H^*((EK_0 \times ET^n) \times_{T^m} P^\mu; \mathbb{Z})$$

According to Lemma 3.8, they are equal to

$$H_{T^m}^*(P^\mu; \mathbb{Z}).$$

Then Theorem 3.10 follows from Theorem 4.8 in [DJ]. \square

Corollary 3.11. For a good contact toric manifold M , we have $H^1(M; \mathbb{Z}) = 0$.

Proof. Consider the spectral sequence of the fiber bundle

$$(3.24) \quad M \hookrightarrow ET^n \times_{T^n} M \rightarrow BT^n$$

If $E_2^{0,1} = H^1(M; \mathbb{Z})$ has torsion, it will be in the kernel of

$$(3.25) \quad d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$$

since $E_2^{2,0} = H^2(BT^n; \mathbb{Z}) = \mathbb{Z}^n$ is free. So the torsion part will live to $E_\infty^{0,1}$. Thus $H^1(ET^n \times_{T^n} M; \mathbb{Z})$ will have torsion and contradicts Theorem 3.10, from which it easily follows that $H^1(ET^n \times_{T^n} M; \mathbb{Z}) = 0$. So $H^1(M; \mathbb{Z})$ is torsion-free.

Theorem 1.1 in [L2] tells us that $\pi_1(M)$ is a finite group. So $H_1(M; \mathbb{Z})$ is finite, and $H^1(M; \mathbb{Z})$ has no free part.

Thus, we may conclude that $H^1(M; \mathbb{Z}) = 0$. \square

4. Cohomology and equivariant cohomology of symplectic toric manifolds

In this section, we recall some old construction and facts about symplectic manifolds. This will help to set up notations for Section 5. In Section 5, we will take $k = n - 1$.

First we will briefly recall the classical Delzant construction of a symplectic toric manifold. Let the convex Delzant polytope be as in (2.3):

$$(4.1) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

Then define a map

$$(4.2) \quad \pi_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^k$$

by sending the i^{th} standard basis vector e_i of \mathbb{Z}^m to \tilde{v}_i . This induces a map

$$(4.3) \quad \pi_{\mathbb{R}} : \mathbb{R}^m \rightarrow \mathbb{R}^k$$

by tensoring with \mathbb{R} . We then get a map

$$(4.4) \quad \pi_T : T^m = \mathbb{R}^m / \mathbb{Z}^m \rightarrow T^k = \mathbb{R}^k / \mathbb{Z}^k$$

The subscripts $\mathbb{Z}, \mathbb{R}, T$ will be omitted sometimes when it will not cause confusion.

Letting $K = \ker(\pi_T)$, we have a short exact sequence of groups

$$(4.5) \quad 1 \longrightarrow K \xrightarrow{i} T^m \xrightarrow{\pi_T} T^k \longrightarrow 1.$$

This induces maps on Lie algebras

$$(4.6) \quad 0 \longrightarrow \mathfrak{k} \xrightarrow{i_*} \mathfrak{t}^m \xrightarrow{\pi_*} \mathfrak{t}^k \longrightarrow 0,$$

and maps on the duals of Lie algebras

$$(4.7) \quad 0 \longleftarrow \mathfrak{k}^* \xleftarrow{i^*} (\mathfrak{t}^m)^* \xleftarrow{\pi^*} (\mathfrak{t}^k)^* \longleftarrow 0.$$

Let $u : \mathbb{C}^m \rightarrow (\mathfrak{t}^m)^*$ be defined by

$$(4.8) \quad u(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$$

Then

$$(4.9) \quad N = ((i^* \circ u)^{-1}(i^*(-\eta)))/K$$

is the symplectic toric manifold associated to the polytope P , where $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m = (\mathfrak{t}^m)^*$.

The standard action of T^m on \mathbb{C}^m restricts to a T^m -action on $(i^* \circ u)^{-1}(i^*(-\eta))$, and hence a K -action on $(i^* \circ u)^{-1}(i^*(-\eta))$. It induces a $T^k \cong T^m/K$ action on $N = ((i^* \circ u)^{-1}(i^*(-\eta)))/K$.

In other words, the action of $t_k \in T^k$ on N is induced by the standard action of any element in $\pi^{-1}(t_k)$ on $(i^* \circ u)^{-1}(i^*(-\eta))$. This action is Hamiltonian with moment map

$$(4.10) \quad \nu : N \rightarrow (\mathfrak{t}^k)^*$$

$$(4.11) \quad [z_1, \dots, z_m] \mapsto (\pi^*)^{-1}(u(z_1, \dots, z_m) + \eta),$$

using that π^* is injective. The image of ν is P . It is easy to check from the construction that P is the orbit space of the T^k -action on N .

Following the same line as Proposition 3.4, we have the following.

Proposition 4.1. *With the same notation as in Example 2.3, $P^{\tilde{\lambda}}$ is T^k -equivariantly homeomorphic to N , the symplectic manifold associated to P .*

The following diagram was used in [DJ] to compute $H^*(N; \mathbb{Z}) = H^*((T^k \times P)/\tilde{\Delta}; \mathbb{Z})$.

$$(4.12) \quad N \xrightarrow{f_1} ET^k \times_{T^k} N \xleftarrow{f_2} ET^m \times_{T^m} ((T^m \times P)/\Omega) \xrightarrow{f_3} ET^m \times_{T^m} pt.$$

In this diagram, ET^m is taken to be $EK \times ET^k$. Since K is connected, there is a map $r : T^m \rightarrow K$, such that $r|_K$ is identity map. A group element $t_m \in T^m$ acts on $ET^m = EK \times ET^k$ via the diagonal action of $(r(t_m), \pi(t_m))$.

The map f_1 is inclusion of fiber, f_3 is the projection onto the first factor. The map f_2 is given by $[(x, y), [t_m, p]] \mapsto [y, [\pi(t_m), p]]$, where $x \in EK, y \in ET^k, t_m \in T^m, p \in P$.

Theorem 4.2. [DJ] *Taking cohomology of diagram (4.12) allows us to compute the equivariant cohomology of the symplectic toric manifold*

$$(4.13) \quad H_{T^k}^*(N; \mathbb{Z}) = H^*(ET^k \times_{T^k} (T^k \times P)/\tilde{\Delta}; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m]/\mathcal{I},$$

where $x_i \in H_{T^k}^2(N; \mathbb{Z})$, $1 \leq i \leq m$, are images of generators of $H^*(ET^m \times_{T^m} pt, \mathbb{Z})$. And the ideal \mathcal{I} was defined in Definition 3.9.

Definition 4.3. *Suppose $\tilde{v}_j = (v_{j1}, v_{j2}, \dots, v_{jk})$, for $1 \leq j \leq m$. Let*

$$(4.14) \quad J_i = \sum_{j=1}^m v_{ji} x_j,$$

for $1 \leq i \leq k$. We define

$$(4.15) \quad \mathcal{J} = \langle J_1, J_2, \dots, J_k \rangle$$

to be the ideal in $\mathbb{Q}[x_1, x_2, \dots, x_m]$ generated by the linear terms $\{J_1, J_2, \dots, J_k\}$.

Theorem 4.4. [DJ] *Taking cohomology of diagram (4.12) allows us to compute the singular cohomology of symplectic toric manifold.*

$$(4.16) \quad H^*(N; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m] / \langle \mathcal{I}, \mathcal{J} \rangle,$$

where $x_i \in H^2(N; \mathbb{Z})$, for $1 \leq i \leq m$.

If we denote by T^r the subgroup of $T^k = (S^1)^k$ that is the last r copies of S^1 , then the T^k action on N restricts to a T^r action on N . We may compute the T^r -equivariant cohomology of N as an easy corollary of the Theorem 4.2 and Theorem 4.4.

Corollary 4.5. $H_{T^r}^*(N; \mathbb{Z}) \simeq \mathbb{Z}[x_1, x_2, \dots, x_m] / \langle \mathcal{I}, J_1, J_2, \dots, J_{k-r} \rangle$

Proof. The spectral sequence of the singular cohomology with \mathbb{Z} coefficients of the fiber bundle

$$N \hookrightarrow ET^r \times_{T^r} N \rightarrow BT^r$$

degenerates at the E^2 term, since both N and BT^r have cohomology only in even degrees. Thus $ET^r \times_{T^r} N$ also has cohomology only in even degrees.

Denote by T^{k-r} the subgroup of T^k that is the product of the first $k-r$ copies of S^1 in $T^k = (S^1)^k$.

We complete the proof by applying the argument in Theorem 4.14 in [DJ] to the fiber bundle

$$ET^r \times_{T^r} N \hookrightarrow ET^{k-r} \times_{T^{k-r}} (ET^r \times_{T^r} N) \rightarrow BT^{k-r}$$

and also noticing that $ET^{k-r} \times_{T^{k-r}} (ET^r \times_{T^r} N) = ET^k \times_{T^k} N$. \square

5. The singular cohomology of good contact toric manifolds

Given a strictly convex good cone

$$(5.1) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\},$$

there is a good contact toric manifold M associated to it. As we proved in Proposition 3.4, M is T^n -equivariantly homeomorphic to P^λ , which was defined in Example 2.2.

To make computations easier, we first move $C_0 = C \setminus \{\vec{0}\}$ into the upper half space $U\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ using a transformation in $SL(n; \mathbb{Z})$, where $SL(n; \mathbb{Z})$ is naturally included into $SL(n; \mathbb{R})$ as a subgroup. We will show in Proposition 5.3 that we can always do this. This will not change the homeomorphism type of the good contact toric manifolds associated to the cone.

Remark 5.1. The good contact toric manifolds associated to two cones that differ by a transformation in $SL(n; \mathbb{Z})$ are contactomorphic. This fact was not stated in [L1], but it easily follows from the classification theorem there.

Lemma 5.2. *The following conditions are equivalent:*

- (1) $C_0 \subset U\mathbb{R}^n$, where $U\mathbb{R}^n$ stands for the ‘upper half space’: $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.
- (2) $(0, 0, \dots, 0, 1) \in (C^\vee)^\circ$, where $C^\vee = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0, \forall x \in C\}$, and $(C^\vee)^\circ$ is its interior $\{y \in \mathbb{R}^n : \langle y, x \rangle > 0, \forall x \in C_0\}$.
- (3) $(0, 0, \dots, 0, 1)$ can be expressed as a linear combination of $\{v_i, 1 \leq i \leq m\}$ with positive coefficients.

Proof. Denote $(0, 0, \dots, 0, 1)$ by \vec{a} .

(1) \Rightarrow (2): For any $x \in C_0$, $\langle \vec{a}, x \rangle = x_n > 0$.

(2) \Rightarrow (3): The dual cone C^\vee is spanned by rays along v_1, v_2, \dots, v_m . So any vector in the interior of it can be expressed as linear combination of $\{v_i, 1 \leq i \leq m\}$ with positive coefficients.

(3) \Rightarrow (1): For any $x \in C$, we have $\langle x, v_i \rangle \geq 0$, for $1 \leq i \leq m$, with all equalities if and only if $x = \vec{0}$. Suppose

$$\vec{a} = \sum_{i=1}^m k_i v_i, \text{ with } k_i > 0 \ \forall i.$$

Then

$$x_n = \langle x, \vec{a} \rangle = \sum_{i=1}^m k_i \langle x, v_i \rangle \geq 0.$$

The equality holds if and only if $x = \vec{0}$. So for any $x \in C_0$, $x_n > 0$, i.e., $C_0 \subset U\mathbb{R}^n$. \square

Proposition 5.3. *There is an element B in $SL(n; \mathbb{Z})$ so that $B(C_0)$ is in $U\mathbb{R}^n$.*

Proof. Let $v = \sum_{i=1}^m v_i$ and u be the primitive vector in the direction of v . Suppose

$$u = \frac{1}{k}v, \text{ for } k \in \mathbb{Z}_{>0}.$$

Since u is primitive, there exists $D \in SL(n, \mathbb{Z})$, such that, $D(u) = (0, 0, \dots, 0, 1)$.

Define a transformation B of \mathbb{R}^n by:

$$(5.2) \quad x \mapsto Bx = D^T x,$$

where the matrix for D^T is the transpose of the matrix for D . Since $\det D^T = \det D = 1$, we still have $B \in SL(n; \mathbb{Z})$.

Under this transformation B , the cone $C = \bigcup_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$ is taken to

$$(5.3) \quad B(C) = \bigcup_{i=1}^m \{x \in \mathbb{R}^n : \langle x, D(v_i) \rangle \geq 0\}.$$

So the normal vectors to the facets of the new cone $B(C)$ are $\{D(v_i) : 1 \leq i \leq m\}$. Moreover,

$$(5.4) \quad (0, 0, \dots, 0, 1) = D(u) = \sum_{i=1}^m \frac{1}{k} D(v_i).$$

Finally, using Lemma 5.2, we see that the new cone $B(C_0)$ is in $U\mathbb{R}^n$. \square

So without loss of generality, we may now assume that $C_0 \subset U\mathbb{R}^n$. By intersecting C with the hyperplane:

$$(5.5) \quad H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\},$$

we get a polytope, which we will denote by P .

The intersection of facet F_i with H is given by

$$\tilde{F}_i := \left\{ (x_1, \dots, x_{n-1}, 1) \in \mathbb{R}^n : \sum_{j=1}^{n-1} x_j v_{ij} + v_{in} \geq 0 \right\}.$$

Let $\mathcal{F} = \{\tilde{F}_i : 1 \leq i \leq m\}$.

If we identify H with \mathbb{R}^{n-1} via

$$(x_1, \dots, x_{n-1}, 1) \mapsto (x_1, \dots, x_{n-1}),$$

then P can be thought of as a polytope in \mathbb{R}^{n-1} given by

$$(5.6) \quad \{x \in \mathbb{R}^{n-1} : \langle x, \tilde{v}_i \rangle \geq -v_{in}\},$$

where $\tilde{v}_i = (v_{i1}, \dots, v_{i,n-1})$ is normal to \tilde{F}_i in $H = \mathbb{R}^{n-1}$.

To compute the singular cohomology ring of M with integer coefficients, we need to add the following assumption.

Smoothness Criterion: The polytope P is a Delzant polytope in \mathbb{R}^{n-1} and \tilde{v}_i is a primitive vector in \mathbb{Z}^{n-1} .

Remark 5.4. This assumption allows us to stay in the smooth category in the following discussion. In general, P is just a simple rational convex polytope, not necessarily smooth, so the results in this section do not hold in full generality. However, if we only care about rational cohomology, the argument and theorems in this section will still be valid in general. In that case, we need to use toric orbifolds instead of manifolds, and S^1 orbi-bundles instead of “honest” S^1 bundles.

Proposition 5.5. *With the notation as above, the good contact toric manifold M is a principal S^1 bundle over N . The projection map*

is given by

where $t = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in T^n$, and $\tilde{t} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{n-1}}) \in T^{n-1}$ is just t with the last coordinate dropped. The notation $[t, p]$ and $[\tilde{t}, p]$ refers to the equivalence classes of (t, p) and (\tilde{t}, p) respectively.

$$(5.7) \quad S^1 = \{(1, 1, \dots, 1, e^{i\theta_n}) \in T^n : \theta_n \in \mathbb{R}\}$$

The quotient space is

Therefore N is the orbit space, and from the construction, we can see the map from the space M to N just drops the last coordinate of t . \square

$$\begin{array}{ccccccc}
 & H^{2k-1}(N; \mathbb{Z}) & \xrightarrow{\cup e} & H^{2k+1}(N; \mathbb{Z}) & \xrightarrow{d^*} & H^{2k+1}(M; \mathbb{Z}) & \xrightarrow{d_*} & H^{2k}(N; \mathbb{Z}) \\
 & & & & & \searrow \cup e & & \\
 & & & & & & & \\
 & H^{2k+2}(N; \mathbb{Z}) & \xrightarrow{d^*} & H^{2k+2}(M; \mathbb{Z}) & \xrightarrow{d_*} & H^{2k+1}(N; \mathbb{Z}) & \xrightarrow{\cup e} & H^{2k+3}(N; \mathbb{Z}) \longrightarrow
 \end{array}$$
$$(5.9) \quad d_*(d^*\alpha \cup \beta) = \alpha \cup d_*(\beta),$$

for $\alpha \in H^*(N; \mathbb{Z})$ and $\beta \in H^*(M; \mathbb{Z})$.

Now the odd degree cohomology of N vanishes and the long exact sequence breaks down to short exact sequences

$$(5.10) \quad 0 \longrightarrow H^{2k+1}(M; \mathbb{Z}) \xrightarrow{d_*} H^{2k}(N; \mathbb{Z}) \xrightarrow{\cup e} H^{2k+2}(N; \mathbb{Z}) \xrightarrow{d^*} H^{2k+2}(M; \mathbb{Z}) \longrightarrow 0.$$

We will describe the cohomology ring of M in terms of the even part and the odd part.

Definition 5.6. Define the even part of $H^*(M; \mathbb{Z})$ as

$$(5.11) \quad H^{even}(M; \mathbb{Z}) =: \{\alpha \in H^k(M; \mathbb{Z}) : k \text{ is an even integer}\}.$$

Define the odd part of $H^*(M; \mathbb{Z})$ as

$$(5.12) \quad H^{odd}(M; \mathbb{Z}) =: \{\alpha \in H^k(M; \mathbb{Z}) : k \text{ is an odd integer}\}.$$

Remark 5.7. It's easy to see that $H^{even}(M; \mathbb{Z})$ is a subring of $H^*(M; \mathbb{Z})$, while $H^{odd}(M; \mathbb{Z})$ is a module over $H^{even}(M; \mathbb{Z})$.

Put together the exact sequences (5.10) of different degrees, we get the following exact sequence:

$$(5.13) \quad 0 \longrightarrow H^{odd}(M; \mathbb{Z}) \xrightarrow{d_*} H^*(N; \mathbb{Z}) \xrightarrow{\cup e} H^*(N; \mathbb{Z}) \xrightarrow{d^*} H^{even}(M; \mathbb{Z}) \longrightarrow 0.$$

From this exact sequence, we can easily prove the following.

Proposition 5.8. Denote by ρ the map

$$\cup e : H^*(N; \mathbb{Z}) \rightarrow H^*(N; \mathbb{Z}).$$

Then

$$(5.14) \quad H^{even}(M; \mathbb{Z}) \simeq \operatorname{coker} \rho = H^*(N; \mathbb{Z}) / \langle e \rangle,$$

where $\langle e \rangle$ stands for the ideal of $H^*(N; \mathbb{Z})$ generated by the Euler class e . This is a ring isomorphism, and it preserves the degree of the cohomology classes since it is induced by the ring map d^* .

Moreover,

$$(5.15) \quad H^{odd}(M; \mathbb{Z}) \simeq \ker \rho = \operatorname{Ann}(e),$$

where $\operatorname{Ann}(e)$ denotes the annihilator of e in $H^*(N; \mathbb{Z})$. This isomorphism maps a cohomology class in $H^{2k+1}(M; \mathbb{Z})$ to a cohomology class in $H^{2k}(N; \mathbb{Z})$, lowering the degree by 1. It is an $H^{even}(M; \mathbb{Z})$ -module isomorphism in the following sense.

The odd cohomology $H^{odd}(M; \mathbb{Z})$ is an $H^{even}(M; \mathbb{Z})$ -module. Moreover, $\ker \rho = \operatorname{Ann}(e)$ is an $\operatorname{coker} \rho = H^*(N; \mathbb{Z}) / \langle e \rangle$ -module in the natural way, so it is also an $H^{even}(M; \mathbb{Z})$ -module via the identification of $\operatorname{coker} \rho$ and $H^{even}(M; \mathbb{Z})$ given in (5.14). The isomorphism in (5.15) respects these module structures.

Proof. The isomorphism (5.14) follows from the exactness of (5.13) at the second $H^*(N; \mathbb{Z})$ and $H^{even}(M; \mathbb{Z})$ and also the fact that $d^* : H^*(N; \mathbb{Z}) \rightarrow H^{even}(M; \mathbb{Z})$ is a ring homomorphism.

As for (5.15), first notice that it follows from (5.9) that the map

$$(5.16) \quad d_* : H^{odd}(M; \mathbb{Z}) \rightarrow H^*(N; \mathbb{Z})$$

is a $H^*(N; \mathbb{Z})$ -module homomorphism. Exactness of (5.13) at $H^{odd}(M; \mathbb{Z})$ and the first $H^{2k}(N; \mathbb{Z})$ shows that

$$(5.17) \quad d_* : H^{odd}(M; \mathbb{Z}) \rightarrow \ker \rho$$

is an $H^*(N; \mathbb{Z})$ -module isomorphism, hence an $H^*(N; \mathbb{Z})/\langle e \rangle$ -module isomorphism. \square

So to compute the cohomology of M , we only need to describe the Euler class e explicitly as a polynomial in x_1, x_2, \dots, x_m , where x_i 's are generators of $H^*(N; \mathbb{Z})$, as in Theorem 4.4.

Definition 5.9. Define $\tilde{\pi}_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^{n-1}$ by mapping the i^{th} standard basis vector e_i of \mathbb{Z}^m to \tilde{v}_i . This induces a map

$$(5.18) \quad \tilde{\pi} : T^m \rightarrow T^{n-1}.$$

in the same way we defined π_T in (4.4). Let $\tilde{K} = \ker \tilde{\pi}$. Since P is Delzant and \tilde{v}_i is primitive, \tilde{K} is connected, and there exists a splitting $\tilde{r} : T^m \rightarrow \tilde{K}$ such that $\tilde{r}|_{\tilde{K}} = id_{\tilde{K}}$.

We will use the following diagram to compute the Euler class e .

$$\begin{array}{ccccccc} M & & L_1 = ET^{n-1} \times_{T^{n-1}} M & & L_2 & & L_3 = ET^m \times_{T^m} S^1 \\ d \downarrow & & d_1 \downarrow & & d_2 \downarrow & & d_3 \downarrow \\ N & \xrightarrow{f_1} & ET^{n-1} \times_{T^{n-1}} N & \xleftarrow{f_2} & (E\tilde{K} \times ET^{n-1}) \times_{T^m} P^\mu & \xrightarrow{f_3} & ET^m \times_{T^m} pt \end{array}$$

The base spaces and the maps are virtually the same as that of (4.12). An element $t_m \in T^m$ acts on $E\tilde{K} \times ET^{n-1}$ via the diagonal action of $(\tilde{r}(t_m), \tilde{\pi}(t_m))$. We now turn to the top line. On L_1 , $T^{n-1} \circlearrowleft M = (T^n \times P)/\Delta$ by multiplication of T^{n-1} on the first $n-1$ coordinates of T^n . On L_3 , $T^m \circlearrowleft S^1$ with weight $-(v_{1n}, v_{2n}, \dots, v_{mn})$. We will use \vec{v}^l to denote the vector $(v_{1n}, v_{2n}, \dots, v_{mn})$. Finally, L_2 is defined as the pull-back $f_2^*(L_1)$.

Lemma 5.10. As principal- S^1 -bundles over N ,

$$(5.19) \quad f_1^*(L_1) = M.$$

As principal S^1 -bundles over $(E\tilde{K} \times ET^{n-1}) \times_{T^m} P^\mu$,

$$(5.20) \quad L_2 = f_2^* L_1 = f_3^* L_3.$$

Proof. The map f_1 is an inclusion, so the pull back of L_1 by f_1 is just by restriction. Then the equation (5.19) follows from the definition.

To show (5.20), it's enough if we can construct an S^1 -equivariant map from $f_3^* L_3$ to $f_2^* L_1$, which lifts the identity map of the base space $(E\tilde{K} \times ET^{n-1}) \times_{T^m} P^\mu$.

Let $q = [x, y, [e^{i\vec{\theta}}, p]]$ be a point in the base space $(E\tilde{K} \times ET^{n-1}) \times_{T^m} P^\mu$, where $x \in E\tilde{K}$, $y \in ET^{n-1}$, $\vec{\theta} = (\theta_1, \dots, \theta_m)$, $e^{i\vec{\theta}} = (e^{i\theta_1}, \dots, e^{i\theta_m}) \in T^m$, and $p \in P$.

The fiber of $f_3^* L_3$ over the point q is, by the definition of pull-back, the fiber of L_3 over the point $f_3(q) = [(x, y), pt]$, namely

$$(5.21) \quad f_3^* L_3|_q = \{[(x, y), e^{i\beta}] : \beta \in \mathbb{R}\}.$$

The fiber of $f_2^* L_1$ over the point q is, by the definition of pull-back, the fiber of L_1 over the point $f_2(q) = [y, [\tilde{\pi}(e^{i\vec{\theta}}), p]]$, namely

$$(5.22) \quad f_2^* L_1|_q = \{[y, [(\tilde{\pi}(e^{i\vec{\theta}}), e^{i\alpha}), p]] : \alpha \in \mathbb{R}\}.$$

Define a map from $f_3^* L_3|_q$ to $f_2^* L_1|_q$ by sending

$$(5.23) \quad [x, y, e^{i\beta}] \mapsto [y, [(\tilde{\pi}(e^{i\vec{\theta}}), e^{i\beta+i\langle \vec{v}^l, \vec{\theta} \rangle}), p]].$$

We call this map s_q . To show s_q is well-defined, we need to show the definition is independent of the choice of representative of q .

First, assume $e^{i\vec{\phi}} \in T^m$, and so

$$(5.24) \quad [x, y, [e^{i\vec{\theta}}, p]] = [\tilde{r}(e^{i\vec{\phi}})x, \tilde{\pi}(e^{i\vec{\phi}})y, [e^{i(\vec{\theta}+\vec{\phi})}, p]].$$

Using this representative of q , then

$$\begin{aligned} s_q([x, y, e^{i\beta}]) &= s_q([\tilde{r}(e^{i\vec{\phi}})x, \tilde{\pi}(e^{i\vec{\phi}})y, e^{i(\beta+\langle -\vec{v}^l, \vec{\phi} \rangle)}]) \\ &= [\tilde{\pi}(e^{i\vec{\phi}})y, [(\tilde{\pi}(e^{i(\vec{\theta}+\vec{\phi})}), e^{i\beta-i\langle \vec{v}^l, \vec{\phi} \rangle+i\langle \vec{v}^l, (\vec{\theta}+\vec{\phi}) \rangle}), p]] \\ &= [\tilde{\pi}(e^{i\vec{\phi}})y, [(\tilde{\pi}(e^{i\vec{\phi}})\tilde{\pi}(e^{i\vec{\theta}}), e^{i\beta+i\langle \vec{v}^l, \vec{\theta} \rangle}), p]]. \end{aligned}$$

This equals to the RHS of (5.23) by the definition of L_1 as given before Lemma 5.10.

Second, choose a different representative by letting $e^{i\vec{\delta}} \in \Omega_p$, and then

$$(5.25) \quad [x, y, [e^{i\vec{\theta}}, p]] = [x, y, [e^{i\vec{\delta}}e^{i\vec{\theta}}, p]].$$

Using this representative of q , we have:

$$\begin{aligned} s_q([(x, y), e^{i\beta}]) &= [y, [(\tilde{\pi}(e^{i\vec{\delta}}e^{i\vec{\theta}}), e^{i\beta+i\langle \vec{v}^l, \vec{\delta}+\vec{\theta} \rangle}), p]] \\ &= [y, [(\tilde{\pi}(e^{i\vec{\delta}}) \cdot \tilde{\pi}(e^{i\vec{\theta}}), e^{i\langle \vec{v}^l, \vec{\delta} \rangle} \cdot e^{i\beta+i\langle \vec{v}^l, \vec{\theta} \rangle}), p]]. \end{aligned}$$

This also equals to the RHS of (5.23) since

$$(5.26) \quad (\tilde{\pi}(e^{i\vec{\delta}}), e^{i\langle \vec{v}^l, \vec{\delta} \rangle}) \in \Delta_p,$$

which is true by definition of $\Delta, \tilde{\Delta}$ and \vec{v}^l .

So the map s_q defined by (5.23) is well-defined. It's obviously S^1 -equivariant. \square

Since the Euler class of L_3 is $\sum_{i=1}^m v_{in} x_i$, using Lemma 5.10 and the naturality of Euler classes, we conclude

$$(5.27) \quad e = \sum_{i=1}^m v_{in} x_i.$$

Definition 5.11. *We define linear forms*

$$(5.28) \quad J_k = \sum_{i=1}^m v_{ik} x_i, 1 \leq k \leq n,$$

and the ideals

$$(5.29) \quad \mathcal{J} = \langle J_1, \dots, J_{n-1}, J_n \rangle$$

and

$$(5.30) \quad \tilde{\mathcal{J}} = \langle J_1, \dots, J_{n-1} \rangle,$$

where $\langle S \rangle$ denotes the ideal in $\mathbb{Z}[x_1, \dots, x_m]$ generated by the elements of S .

Remark 5.12. Notice that by definition we have $e = J_n$.

Combining Theorem 4.4, Proposition 5.8 and (5.27), we have proved the following:

Theorem 5.13. *Assume*

$$(5.31) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

is a strictly convex good cone and M is the good contact toric manifold associated with it. Further assume that $C_0 = C \setminus \{\vec{0}\} \subset U\mathbb{R}^n$ and the smoothness criterion for C holds. Let

$$(5.32) \quad \rho : \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle \mathcal{I}, \tilde{\mathcal{J}} \rangle} \rightarrow \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle \mathcal{I}, \tilde{\mathcal{J}} \rangle}$$

be multiplication by J_n . Then:

$$(5.33) \quad H^{\text{even}}(M; \mathbb{Z}) \simeq \text{coker } \rho \simeq \mathbb{Z}[x_1, \dots, x_m] / \langle \mathcal{I}, \mathcal{J} \rangle$$

as rings, where x_i represents a cohomology class of degree two.

Moreover,

$$(5.34) \quad H^{odd}(M; \mathbb{Z}) \simeq \ker \rho$$

as $(H^{even}(M; \mathbb{Z}) \simeq \operatorname{coker} \rho)$ -modules. A homogeneous polynomial of degree k represents a cohomology class of degree $2k + 1$ under this isomorphism.

The next theorem states that half of the Betti numbers of M vanish.

Theorem 5.14. *Under the same assumption for M as in Theorem 5.13, we have*

$$(5.35) \quad H^{2k+1}(M; \mathbb{Z}) = 0$$

for $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$, and

$$(5.36) \quad H^{2k}(M; \mathbb{Q}) = 0$$

for $\{k \in \mathbb{N} : n \leq 2k \leq 2n - 2\}$.

Proof. According to Theorem 6.3 in [Gu2], the cohomology class of the Kähler form on N is given by

$$(5.37) \quad [\omega] = 2\pi \sum_{i=1}^m v_{in} D_i,$$

where D_i denotes the cohomology class in $H^2(N; \mathbb{R})$ that is the Poincaré dual to $(T^{n-1} \times \tilde{F}_i)/\tilde{\Delta}$. According to Proposition 6.3 in the Appendix, $D_i = -x_i$. So

$$(5.38) \quad [\omega] = -2\pi e.$$

Using the Hard Lefschetz Theorem on N , it follows easily from Theorem 5.13 that

$$(5.39) \quad H^{2k+1}(M; \mathbb{Q}) = 0$$

for $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$, and

$$(5.40) \quad H^{2k}(M; \mathbb{Q}) = 0$$

for $\{k \in \mathbb{N} : n \leq 2k \leq 2n - 2\}$. Furthermore, as the cohomology ring of a symplectic toric manifold, the ring $\mathbb{Z}[x_1, x_2, \dots, x_m]/\{\mathcal{I}, \tilde{\mathcal{J}}\}$ in Theorem 5.13 is torsion-free (see [F]). Therefore as an ideal of it, $\ker \rho$ is also torsion-free. Hence

$$(5.41) \quad H^{2k+1}(M; \mathbb{Z}) = 0$$

for $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$. □

Corollary 5.15. *Under the same assumption for M as in Theorem 5.13, the product of two odd-degree cohomology classes of M is zero.*

Proof. This is because of dimension considerations. □

Remark 5.16. Theoretically, Theorem 5.13 already tells us all the information of $H^*(M; \mathbb{Z})$ as an additive group. Perhaps there should be a combinatorial proof for Theorem 5.14, but the author could not find one.

Remark 5.17. Theorem 5.13 tells us what the elements of $H^*(M; \mathbb{Z})$ are, (5.33) tells us how to multiply two even-degree cohomology classes, (5.34) tells us how to multiply an even-degree cohomology class and an odd-degree cohomology class. Corollary 5.15 tells us the product of two odd-degree cohomology classes must be zero. So the ring structure of $H^*(M; \mathbb{Z})$ is now completely determined.

6. Appendix: Several equivalent descriptions of the generators of cohomology rings of symplectic toric manifolds

There is a wide variety of descriptions of the generators of the cohomology ring of a symplectic toric manifold in the literature. In this appendix we list some of them and discuss their relations.

Let P be as in (2.3),

$$(6.1) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

We will use N to denote the symplectic toric manifold associated to P . We will keep our notation from Example 2.3, so $N = P^{\tilde{\lambda}} = (T^k \times P)/\tilde{\Delta}$.

The cohomology in this section are assumed to be integral cohomology.

The generators in Theorem 4.4 of this paper, which we denoted by x_i , is the image of the generators of $H^*(BT^m)$ under the composition:

$$(6.2) \quad H^*(BT^m) \rightarrow H^*(ET^m \times_{T^m} ((T^m \times P)/\Omega)) \xrightarrow{\sim} H^*(ET^k \times_{T^k} ((T^k \times P)/\tilde{\Delta})) \rightarrow H^*(N)$$

We will now give an easier description for these generators, starting by defining a S^1 -bundle over $N = (T^k \times P)/\tilde{\Delta}$.

For any fixed $j \in \{1, 2, \dots, k\}$, define a characteristic map from the set of facets of P to \mathbb{Z}^{k+1} :

$$\begin{aligned} \sigma_j : \mathcal{F} &\rightarrow \mathbb{Z}^{k+1} \\ \tilde{F}_j &\mapsto (v_{j1}, v_{j2}, \dots, v_{jk}, 1); \\ \tilde{F}_i &\mapsto (v_{i1}, v_{i2}, \dots, v_{ik}, 0) \text{ for } i \neq j. \end{aligned}$$

Then P^{σ_j} is a principal S^1 -bundle over $P^{\tilde{\lambda}}$ for similar reasons as in Proposition 5.5.

For the same reason as (5.20) and (5.27), we have the following proposition.

Proposition 6.1. *Each generator x_j in Theorem 4.4 is the Euler class, or equivalently, the first Chern class, of the principal S^1 -bundle P^{σ_j} over $N = P^{\tilde{\lambda}}$.*

Recall from (4.9), N is the quotient of a set $(i^* \circ u)^{-1}(i^*(-\eta))$, which we will denote by Z , by a torus K . Then Z is a principal- K bundle over N .

In the book [GuSt], the cohomology of a symplectic toric manifold is computed in a very different way (Theorem 9.8.6 of the book). The generators for the cohomology ring are c_1, c_2, \dots, c_m , where (c_1, c_2, \dots, c_m) are the Chern classes of the bundle $Z \rightarrow N$. These c_i 's can also be described in the following way, as illustrated in both Section 2.2 of [Gu] and Section 9.8 of [GuSt].

The torus T^m acts on \mathbb{C}^m in the standard way and \mathbb{C}^m splits

$$(6.3) \quad \mathbb{C}^m = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \dots \oplus \mathbb{C}_m.$$

The torus K , as a subgroup of T^m , also acts on \mathbb{C}^m and preserves this splitting. Then

$$(6.4) \quad (Z \times \mathbb{C}_i)/K \rightarrow Z/K$$

is a complex line bundle over $N = Z/K$, where the action of K on \mathbb{C}_i is by first including K into T^m , then acting with weight $-(0, \dots, 0, 1, 0, \dots, 0)$, where the only 1 is on the i^{th} position. The K action on $Z \times \mathbb{C}_i$ is the diagonal action. Then the first Chern class of this line bundle is c_i .

Proposition 6.2. *The generator c_i is exactly the generator x_i used in Theorem 4.4.*

Proof. Assume without loss of generality that $i = 1$. According to Proposition 6.1, it suffices to show P^{σ_1} and $(Z \times S^1)/K$ are isomorphic principal- S^1 bundles over $P^{\tilde{\lambda}} = Z/K$, where the action of K on S^1 is by first including K into T^m , then acting with weight $-(1, 0, 0, \dots, 0)$. The base spaces are identified by Proposition 4.1.

Fixing a splitting $\alpha : T^k \rightarrow T^m$, such that

$$(6.5) \quad \pi \circ \alpha = id_{T^k}.$$

Define

$$\begin{aligned} f : P^{\sigma_1} &\rightarrow (Z \times S^1)/K \\ [t_k, e^{i\theta}, p] &\mapsto [\alpha(t_k).u_0^{-1}(\pi^*(p) - \eta), \alpha(t_k).e^{i\theta}], \end{aligned}$$

where $t_k \in T^k$, $\theta \in \mathbb{R}$, $p \in P$, and $u_0 : (\mathbb{R}_{\geq 0})^m \rightarrow (\mathbb{R}_{\geq 0})^m$ is given by $(z_1, \dots, z_m) \mapsto (z_1^2, \dots, z_m^2)$. The verification that this map is well-defined is routine. It is easy to see this map lifts the identity map of the base space, is S^1 -equivariant and non-trivial on each fiber. So P^{σ_1} and $(Z \times S^1)/K$ are isomorphic, hence $c_1 = x_1$. \square

The set $(T^k \times \tilde{F}_i)/\tilde{\Delta}$ is a submanifold of N . It is the pre-image of \tilde{F}_i under the moment map ν as defined in (4.10). Its Poincare dual, by definition, is a cohomology class in $H^2(N; \mathbb{R})$. We denote it by D_i .

Proposition 6.3. *The class D_i equals $-c_i = -x_i$.*

Proof. Without loss of generality, assume $i = 1$. It is obvious that $-c_1$ is the first Chern class of the complex line bundle

$$(6.6) \quad (Z \times \mathbb{C})/K \rightarrow Z/K,$$

where K , as a subgroup of T^m , acts on \mathbb{C} with weight $(1, 0, 0, \dots, 0)$. According to Proposition 12.8 in [BT], it suffices to find a transversal section of this line bundle, such that the zero locus of the section is exactly $\nu^{-1}(\tilde{F}_1)$. Define a section

$$\begin{aligned} s : \quad Z/K &\rightarrow (Z \times \mathbb{C})/K \\ [z_1, z_2, \dots, z_m] &\mapsto [(z_1, z_2, \dots, z_m), z_1], \end{aligned}$$

noticing that Z is a subset of \mathbb{C}^m . It is easy to see this map is well-defined, and it is straightforward to show that

$$(6.7) \quad z_1 = 0 \Leftrightarrow \nu([z_1, \dots, z_m]) \in \tilde{F}_1.$$

Finally, to see it is transversal to the zero section, simply notice it is holomorphic and obviously not tangent to the zero section along the zero locus. \square

In [TW], as a corollary of a more general theorem, there is yet another way of computing the cohomology ring of a symplectic toric manifold. To describe the generators there, we draw a diagram first:

$$(6.8) \quad (EK \times ET^k) \times_{T^m} \mathbb{C}^m \xleftarrow{g_1} EK \times_K \mathbb{C}^m \xleftarrow{g_2} EK \times_K Z \xrightarrow{g_3} Z/K.$$

In the diagram, g_1 and g_2 are just inclusion maps, g_3 is a fiber bundle with fiber EK . The group T^m acts on $EK \times ET^k$ as explained in Section 4. Now

$$(6.9) \quad H^*((EK \times ET^k) \times_{T^m} \mathbb{C}^m) = \mathbb{Z}[y_1, \dots, y_m],$$

where y_i is the Chern class of the principal S^1 -bundle

$$(6.10) \quad (EK \times ET^k) \times_{T^m} (\mathbb{C}^m \times S^1) \rightarrow (EK \times ET^k) \times_{T^m} \mathbb{C}^m,$$

where T^m acts on S^1 with weight $-(0, 0, \dots, 0, 1, 0, \dots, 0)$, where the only 1 is on the i^{th} position. In Theorem 7 of [TW], the generators of the cohomology ring of a symplectic toric manifold are the images of these y_i 's under the composed map

$$(6.11) \quad H^*((EK \times ET^k) \times_{T^m} \mathbb{C}^m) \xrightarrow{g_1^*} H^*(EK \times_K \mathbb{C}^m) \xrightarrow{g_2^*} H^*(EK \times_K Z) \xrightarrow{(g_3^*)^{-1}} H^*(Z/K).$$

It follows easily from the naturality of Chern class that $g_2^* g_1^*(y_i) = g_3^*(c_i)$, whence we may conclude our final proposition.

Proposition 6.4. *The generators for the cohomology ring of a symplectic toric manifold in Theorem 7 in [TW] are the same as the ones used in Theorem 4.4 in this paper.*

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